EXHIBIT 11
Case 1597

EXCLUSIVE

How to use the EXPONENTIAL INTEGRAL

No more difficult than logarithms, Ei-functions solve problems of well interference and effective permeability. Use is demonstrated, abbreviated tables included with examples and simplified curve

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THE Exponential Integral, abbreviated Ei, appears in a mathematical solution of problems involving the flow of a single phase compressible fluid through a homogeneous, infinite porous medium under non steady state conditions. This solution is based on two assumptions: (1) the porous medium has cylindrical symmetry, and (2) the well radius is very small compared to the effective radius of the porous medium.

This solution has been widely used in recent years, and provides a more realistic answer than is provided by the so-called steady-state solutions.

Exponential Integrals are no more difficult to use than logarithms, or the trignometric functions. There is nothing mysterious or difficult about them. The Exponential Integral arises in the solution of the differential equation for the flow of a single liquid ph[illegible] constant compressibility in a homogeneous porou[illegible]m. The resulting equation for the case of a constant p[illegible]on rate in a system having cylindrical symmetry is:

$$(1)\quad p_e - p(r,T) = \frac{q_o\,u\,B}{0.0141\,kh}\left[-\,Ei\left(-\frac{948.4\,cufr^2}{kT}\right)\right]$$

where p_e = formation pressure in psi.
$p(r,T)$ = pressure, in psi, at a radial distance, r, from the well at time T.
T = time in hours after opening up the well.
r = radial distance in feet from the well.
q_o = production rate in stock tank barrels per day.
u = viscosity, centipoises.
B = formation volume factor (dimensionless).
k = permeability in millidarcys.
h = thickness of producing formation in feet.
c = compressibility of the reservoir fluid in $\frac{1}{\text{psi}}$.
f = porosity, fractional.

The Exponential Integral in Equation (1) is the term in the brackets.

$$\left[-\,Ei\left(-\frac{948.4\,cufr^2}{kT}\right)\right]$$

The Exponential Integral is defined as

$$(2)\quad -Ei(-x) = \int_x^{\infty}\frac{e^{-v}}{v}\,dv$$

In this definition v is a dummy variable and disappears upon integration and substitution of the limits. $-Ei(-x)$ is thus a function of x only. The Exponential Integral can also be expressed by means of infinite series, thus:

$$(3)\quad -Ei(-x) = e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots\right)$$

$$(4)\quad -Ei(-x) = -0.5772 - 2.303\log_{10}x + x - \frac{x^2}{4} + \frac{x^3}{18} - \cdots \frac{x^n}{n.n!}$$

The form given in Equation (4) is particularly useful since it permits a simple evaluation of the Ei function for values of x outside the range of tables, or if no tables are available. It is also used to determine the range of x over which the logarithmic approximation may be used.

Example 1. Calculate $-Ei(-0.25)$. By Equation (4).

$$-Ei(-0.25) = -0.5772 - 2.303\log_{10}(0.25) + 0.25 - \frac{(0.25)^2}{4} + \frac{(0.25)^3}{18} - \cdots$$
$$= -0.5772 - 2.303(-0.6021) + 0.25 - \frac{0.0625}{4} + \frac{0.015625}{18} - \cdots$$
$$= -0.5772 + 1.3866 + 0.25 - 0.0156 + 0.0008$$
$$= -0.5928 + 1.6366$$
$$= 1.0438$$

The value 1.0438 may be rounded to the value 1.044 which is usually sufficient for most reservoir problems. It will be noted that the last term in the series (0.0008) was not used. A set of tables, Ref (4), gives the value of 1.0443. The difference between 1.0443 and 1.0438 is 0.0005 which is less than the value of the last term calculated, but not used. This illustrates the rule that for series of the type of Equation (4) the error resulting from omitting all terms after a certain selected one is less than the first term neglected. In this example all terms after the term $\frac{x^2}{4}$ were neglected and the error is shown to be less than the value of the term $\frac{x^3}{18}$, which is (0.0008). Here the term $\frac{x^3}{18}$ is the first of the terms neglected.

Example 2. Assume that it is desirable to use the logarithmic approximation. Assume further that a value of $-Ei(-x)$ accurate to 0.01 is acceptable. In other words, all terms in Equation (4) are to be neglected after the logarithmic term. What is the largest value that x can have in order that the error in $-Ei(-x)$ shall not exceed 0.01? It was shown in the previous example that the error does not exceed the value of the first term neglected. In this case, the first term neglected is x. Hence, if x does not exceed 0.01, then the error in $-Ei(-x)$ resulting from the use of the logarithmic approximation will not exceed 0.01.

Tables of the Ei function are available and are used in the same manner as logarithmic, or trigonometric tables. A condensed table is given in Appendix A. More complete tables may be purchased from the Superintendent of Documents (4), Washington 25, D. C. For most reservoir work, however, the tables given in Appendix A are sufficiently accurate. For rougher work, a graph based on equation (1) has been prepared and is given in Appendix B.

The two previous examples illustrated the method of calculating the value of the function $-Ei(-x)$. Two more examples are presented illustrating the use of the Exponential Integral in Equation (1).

Example 3. This example applies to the problem of well interference. Assume two wells are separated by a distance of 1100 ft. Both wells have been shut-in for a sufficient length of time that the pressure in each is the static reservoir pressure. Also, assume that the common formation in which these wells are completed is homogeneous and continuous. The problem is to calculate how many hours it will take for a pressure drop of 5 psi to occur in well B after well A commences to produce at a rate of 250 stock tank bbl per day.

Assume further, that the following quantities have previously been determined:

$k = 133$ md

$h = 33$ feet

$kh = 4389$ md-ft

$u = 0.38$ cp.

$B = 1.47$

$f = 0.02$

$c = 1.59 \times 10^{-5} \frac{1}{psi}$

The remaining quantities in Equation (1) as previously specified are $r = 1100$ ft, $q_o = 250$ STB/D, $p_o - p(r,T) = \Delta P = 5$ psi

Substitution of these quantities in Equation (1) gives:

$$\frac{5 \times 0.0141 \times 4389}{250 \times 0.38 \times 1.47} = -Ei\left(-\frac{948.4 \times 1.59 \times 10^{-5} \times 0.38 \times 0.02 \times (1100)^2}{33\,T}\right)$$

$$2.218 = -Ei\left(-\frac{4.202}{T}\right)$$

Thus, $\frac{4.202}{T}$ is the x in $-Ei(-x)$ and 2.218 is the value of the Ei-function. The next step is to find the value of x from the tables. The quantity in the body of the table nearest 2.218 is 2.20. Hence, the value of x, to three decimals, read from the table, is 0.065.

$$\frac{4.202}{T} = 0.065$$

$$T = \frac{4.202}{0.065} = 64.6 \text{ hours}$$

Example 4. Two wells, A and B, are separated by a distance of 1100 ft. A well pressure build-up test on A has yielded an effective reservoir productivity of 4400 md-ft (kh). When A is produced at a constant rate of 275 STB/day, a pressure drop of 10 psi is observed at B after 108 hours. Fluid and formation constants are:

$u = 0.40$ cp.

$B = 1.47$

$f = 0.02$

$c = 1.59 \times 10^{-5} \frac{1}{psi}$

What is the effective permeability of the intervening formation?

Substituting these values in Equation (1).

$$\frac{10 \times 0.0141 \times 4400}{275 \times 0.40 \times 1.47} = -Ei\left(-\frac{948.4 \times 1.59 \times 10^{-5} \times 0.4 \times 0.02 \times (1100)^2}{k \times 108}\right)$$

$$4.039 = -Ei\left(-\frac{1.35}{k}\right)$$

$$\frac{1.35}{k} = 0.010 \text{ (from the tables)}$$

$$k = 135. \text{ md. (effective)}$$

Acknowledgment

The author wishes to thank the Management of Phillips Petroleum Company for permission to publish this article.

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APPENDIX A

Table of the Exponential Integral

$f(x) = -Ei(-x)$

$0.00 < x < 0.209$, Interval = 0.001

X	0	1	2	3	4	5	6	7	8	9
0.00	+∞	6.332	5.639	5.235	4.948	4.726	4.545	4.392	4.260	4.142
0.01	4.038	3.944	3.858	3.779	3.705	3.637	3.574	3.514	3.458	3.405
0.02	3.355	3.307	3.261	3.218	3.176	3.137	3.098	3.062	3.026	2.992
0.03	2.959	2.927	2.897	2.867	2.838	2.810	2.783	2.756	2.731	2.706
0.04	2.681	2.658	2.634	2.612	2.590	2.568	2.547	2.527	2.507	2.487
0.05	2.468	2.449	2.431	2.413	2.395	2.377	2.360	2.344	2.327	2.311
0.06	2.295	2.279	2.264	2.249	2.235	2.220	2.206	2.192	2.178	2.164
0.07	2.151	2.138	2.125	2.112	2.099	2.087	2.074	2.062	2.050	2.039
0.08	2.027	2.015	2.004	1.993	1.982	1.971	1.960	1.950	1.939	1.929
0.09	1.919	1.909	1.899	1.889	1.879	1.869	1.860	1.850	1.841	1.832
0.10	1.823	1.814	1.805	1.796	1.788	1.779	1.770	1.762	1.754	1.745
0.11	1.737	1.729	1.721	1.713	1.705	1.697	1.689	1.682	1.674	1.667
0.12	1.660	1.652	1.645	1.638	1.631	1.623	1.616	1.609	1.603	1.596
0.13	1.589	1.582	1.576	1.569	1.562	1.556	1.549	1.543	1.537	1.530
0.14	1.524	1.518	1.512	1.506	1.500	1.494	1.488	1.482	1.476	1.470
0.15	1.464	1.459	1.453	1.447	1.442	1.436	1.431	1.425	1.420	1.415
0.16	1.409	1.404	1.399	1.393	1.388	1.383	1.378	1.373	1.368	1.363
0.17	1.358	1.353	1.348	1.343	1.338	1.333	1.329	1.324	1.319	1.314
0.18	1.310	1.306	1.301	1.296	1.291	1.287	1.282	1.278	1.274	1.269
0.19	1.265	1.261	1.256	1.252	1.248	1.243	1.239	1.235	1.231	1.227
0.20	1.223	1.219	1.215	1.210	1.206	1.202	1.198	1.195	1.191	1.187

X	0	1	2	3	4	5	6	7	8	9
0.0	+∞	4.038	3.355	2.959	2.681	2.468	2.295	2.151	2.027	1.919
0.1	1.823	1.737	1.660	1.589	1.524	1.464	1.409	1.358	1.309	1.265
0.2	1.223	1.183	1.145	1.110	1.076	1.044	1.014	0.985	0.957	0.931
0.3	0.906	0.882	0.858	0.836	0.815	0.794	0.774	0.756	0.737	0.719
0.4	0.702	0.686	0.670	0.655	0.640	0.625	0.611	0.598	0.585	0.572
0.5	0.560	0.548	0.536	0.525	0.514	0.503	0.493	0.483	0.473	0.464
0.6	0.454	0.445	0.437	0.428	0.420	0.412	0.404	0.396	0.388	0.381
0.7	0.374	0.367	0.360	0.353	0.347	0.340	0.334	0.328	0.322	0.316
0.8	0.311	0.306	0.300	0.295	0.289	0.284	0.279	0.274	0.269	0.265
0.9	0.260	0.256	0.251	0.247	0.243	0.239	0.235	0.231	0.227	0.223
1.0	0.219	0.216	0.212	0.209	0.205	0.202	0.198	0.195	0.192	0.189
1.1	0.186	0.183	0.180	0.177	0.174	0.172	0.169	0.166	0.164	0.161
1.2	0.158	0.156	0.153	0.151	0.149	0.146	0.144	0.142	0.140	0.138
1.3	0.135	0.133	0.131	0.129	0.127	0.125	0.124	0.122	0.120	0.118
1.4	0.116	0.114	0.113	0.111	0.109	0.108	0.106	0.105	0.103	0.102
1.5	0.1000	0.0985	0.0971	0.0957	0.0943	0.0929	0.0915	0.0902	0.0889	0.0876
1.6	0.0863	0.0851	0.0838	0.0826	0.0814	0.0802	0.0791	0.0780	0.0768	0.0757
1.7	0.0747	0.0736	0.0725	0.0715	0.0705	0.0695	0.0685	0.0675	0.0666	0.0656
1.8	0.0647	0.0638	0.0629	0.0620	0.0612	0.0603	0.0595	0.0586	0.0578	0.0570
1.9	0.0562	0.0554	0.0546	0.0539	0.0531	0.0524	0.0517	0.0510	0.0503	0.0496
2.0	0.0489	0.0482	0.0476	0.0469	0.0463	0.0456	0.0450	0.0444	0.0438	0.0432

APPENDIX A (Concluded)

Table of Exponential Integral

$f(x) = -Ei(-x)$

$2.0 < x < 10.9$, Interval = 0.1

X	0	1	2	3	4	5	6	7	8	9
2.	4.89×10^{-2}	4.26×10^{-2}	3.72×10^{-2}	3.25×10^{-2}	2.84×10^{-2}	2.49×10^{-2}	2.19×10^{-2}	1.[illegible]	1 [illegible]	1.[illegible]
3.	1.30×10^{-2}	1.15×10^{-2}	1.01×10^{-2}	8.94×10^{-3}	7.89×10^{-3}	6.97×10^{-3}	6.16×10^{-3}	5.45×10^{-3}	4.[illegible]	4.27×10^{-3}
4.	3.78×10^{-3}	3.35×10^{-3}	2.97×10^{-3}	2.64×10^{-3}	2.34×10^{-3}	2.07×10^{-3}	1.94×10^{-3}	1.64×10^{-3}	1.[illegible]	1.[illegible]
5.	1.15×10^{-3}	1.02×10^{-3}	9.08×10^{-4}	8.08×10^{-4}	7.19×10^{-4}	6.41×10^{-4}	5.71×10^{-4}	5.[illegible]	4.[illegible]	4.[illegible]
6.	3.60×10^{-4}	3.21×10^{-4}	2.[illegible]	2.56×10^{-4}	2.28×10^{-4}	2.03×10^{-4}	1.82×10^{-4}	1.[illegible]	1.[illegible]	1.[illegible]
7.	1.15×10^{-4}	1.03×10^{-4}	9.22×10^{-5}	8.24×10^{-5}	7.36×10^{-5}	6.58×10^{-5}	5.90×10^{-5}	5.[illegible]	4.71×10^{-5}	4.21×10^{-5}
8.	3.77×10^{-5}	3.37×10^{-5}	3.02×10^{-5}	2.70×10^{-5}	2.42×10^{-5}	2.18×10^{-5}	1.94×10^{-5}	1.72×10^{-5}	1.[illegible]	1.[illegible]
9.	1.24×10^{-5}	1.11×10^{-5}	9.99×10^{-6}	8.95×10^{-6}	8.02×10^{-6}	7.19×10^{-6}	6.44×10^{-6}	5.77×10^{-6}	5.17×10^{-6}	4.64×10^{-6}
10.	4.16×10^{-6}	3.73×10^{-6}	3.34×10^{-6}	3.00×10^{-6}	2.68×10^{-6}	2.41×10^{-6}	2.16×10^{-6}	1.94×10^{-6}	1 [illegible]	1 [illegible]

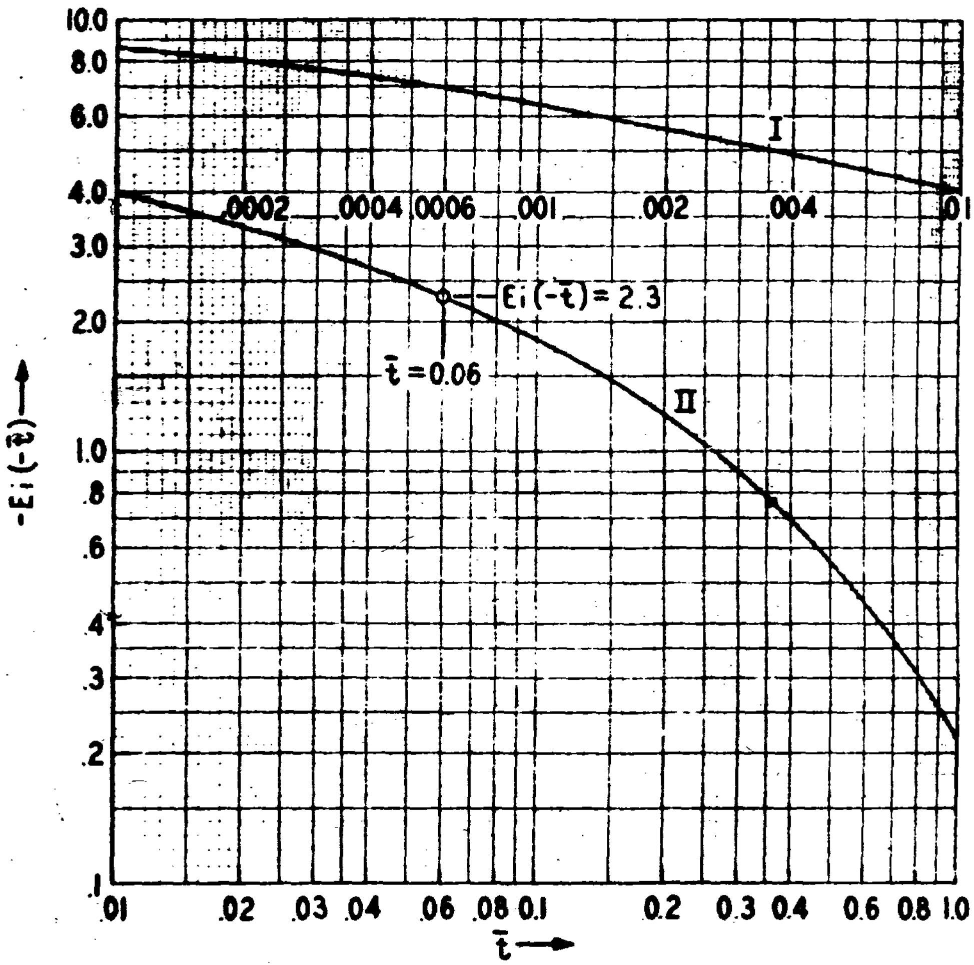

Fig. 1. Graph of the function $f(\bar{t}) = -Ei(-\bar{t})$ can be used when accuracy to two significant figures is adequate.

APPENDIX B

Chart for the Calculation of the Exponential Integral

By rearranging Equation (1) and introducing dimensionless variables, the calculation of Exponential Integrals is simplified. This chart may be used whenever accuracy to two significant figures is adequate for the problem at hand.

Let $\Delta P = p_e - p(r,T)$

$$\bar{P} = \frac{q_o\, u\, B}{0.0141\, k\, h}$$

$$\bar{t} = 948.4 \frac{c\, u\, f\, r^2}{k\, T}$$

Then, Equation (1) may be written

$$\Delta P = \bar{P}\,[-Ei(-\bar{t})]$$

The chart consists of two branches of the Ei-curve plotted on double logarithmic paper. Values of $\bar{t}$ are plotted along the x-axis. Values of $-Ei(-\bar{t})$ are plotted along the Y-axis.

The first branch, marked I, covers the range of $\bar{t}$ from 0.0001 to 0.01; the second branch, marked II, covers the range of $\bar{t}$ from 0.01 to 1.0. Values of $\bar{t}$ are indicated along these curves for convenience in reading. Values of $-Ei(-\bar{t})$ covers the range from 0.22 to 8.6. These are indicated at the left of the chart.

Example 1, Consider the point O_1 $(\bar{t})$ located on Branch II.

$\bar{t} = 0.06$, $-Ei(-\bar{t}) = 2.3$